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LE ANH TUAN

$\mathcal{H}_\infty$ CONTROL PROBLEM FOR SOME CLASSES OF TIME-DELAY SYSTEMS

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Scientific Advisor: Prof. PhD.Sci. Vu Ngoc Phat

Referee 1: Prof. PhD.Sci. Nguyen Huu Du, VNU University of Science - Vietnam National University, Hanoi

Referee 2: Assoc.Prof. PhD. Ha Tien Ngoan, Institute of Mathematics, VAST

Referee 3: Assoc.Prof. PhD. Tran Dinh Ke, Hanoi National University of Education

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INTRODUCTION

1. MOTIVATION

Stability theory is an important branch of the qualitative theory of differential equations that was initiated by the Russian mathematician A.M. Lyapunov in the late nineteenth century. With a long history of more than a century, Lyapunov stability theory is still a very attractive field of mathematics, with more and more important applications being found in mechanics, physics, chemistry, information technology, ecology, environment, etc. (see Gu et al. (2003), Hinrichsen and Pritchard (2010), Kolmanovskii and Myshkis (1999), Krasovskii (1963)).

In addition to stability of solutions, problem of stabilization of the control systems was also taken into consideration, and they began to study the stability of the control systems in the 1960s. On the other hand, in mathematical models (are constructed from technical problems in practice) there are often time delays. These delays are naturally formed, unavoidable in the transmission and processing of data, and they proved that its presence will have an effect on the behaviour and properties of the system, including stability (see Gu et al. (2003), Niculescu (2001)). Therefore, the study of stability and control for delayed systems are problems of applicable practicality that have been studied by many scholars in recent years (see Boyd et al. (1994), Duan and Yu (2013), Fridman (2014), Michiels and Niculescu (2014)).

In addition, processes in practice are often uncertain (with the appearance of system disturbances). These disturbances can occur due to operational errors, due to interactions between components in the system or between different systems. Therefore, it is unable to know exactly all the parameters of the system in the model, or very difficult to apply in practice. As a result, the optimal evaluation of the effect of disturbances on the output of the system (the $\mathcal{H}_\infty$ problem) is a topical problem, studied by many mathematicians and engineers. Different approaches have been developed and a large number of important findings on the $\mathcal{H}_\infty$ control for many delayed systems have been published in recent years (see Petersen et al. (2000), Wu et al. (2010), Xu and Lam (2006), Zhou et al. (1995)). However, many interesting and important open issues in both theory and application remain unresolved, in particular the existing findings for the $\mathcal{H}_\infty$ problem

for many kinds of the delayed control systems are quite modest and needs to be further researched. That is the motivation for us to implement this topic.

2. OVERVIEW OF THE RESEARCH PROBLEMS

The (delayed) neural networks is a special class of functional differential equations, which have been studied extensively for more than two decades by its successful applications in many fields such as associative memory, identification and classification, signal processing, image processing, optimal problems solving, etc. Therefore, class of the first systems is mentioned in the thesis on the $\mathcal{H}_\infty$ control problem is the neural networks with mixed time-varying delays:

$$\begin{aligned}
\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2 \int_{t-k(t)}^{t} c(x(s))ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + Bu(t) + C\omega(t) \\
z(t) &= Ex(t) + Mx(t-h(t)) + Nu(t), \quad t \geqslant 0, \\
x(t) &= \varphi(t), \quad t \in [-d, 0], \quad d = \max\{h_2, k\},
\end{aligned} \tag{1}$$

where $h(t)$, $k(t)$ are delay functions satisfying the condition $0 \leqslant h_1 \leqslant h(t) \leqslant h_2,\ 0 \leqslant k(t) \leqslant k$.

In 2009, the exponential stability problem of the neural networks

$$\dot{x}(t) = -(A+\Delta A(t))x(t)+(W_0+\Delta W_0(t))f(x(t))+(W_1+\Delta W_1(t))f(x(t-h(t)))$$

with the function $h(t)$ is interval time-varying delay with bounded derivative that was considered by Kwon and Park. On exponentially stabilizability problem, the authors Phat, Trinh proposed in 2010 for the neural networks with mixed time-varying delays

$$\dot{x}(t) = -Ax(t)+W_0 f(x(t))+W_1 g(x(t-h(t)))+W_2 \int_{t-k(t)}^{t} c(x(s))ds+Bu(t),$$

where delay functions $h(t)$, $k(t)$ are assumed to satisfy the condition: $0 \leqslant h(t) \leqslant h,\ \dot{h}(t) \leqslant \delta < 1,\ 0 \leqslant k(t) \leqslant k \quad \forall t \geqslant 0$. Soon after, this result was extended to the case where discrete delay $h(t)$ is a continuous function, taking value in a interval by two authors Thuan and Phat (2012). In 2012, Sakthivel et al. considered the $\mathcal{H}_\infty$ control problem for the mixed delay

neural networks (and there is no delay in the observation)

$$\begin{aligned}\dot{x}(t) &= -(A+\Delta A)x(t) + (W_0+\Delta W_0)f(x(t)) + (W_1+\Delta W_1)g(x(t-h(t)))\\ &\quad + (W_2+\Delta W_2)\int_{t-k(t)}^{t} c(x(s))ds + u(t) + (C+\Delta C)\omega(t),\\ z(t) &= Ex(t),\end{aligned}$$

with delay functions $h(t)$, $k(t)$ satisfy: $0 \leqslant h(t) \leqslant h,\ \dot{h}(t) \leqslant \delta,\ 0 \leqslant k(t) \leqslant k \quad \forall t \geqslant 0$. In this paper, the authors obtained a asymptotically stabilizability and $\mathcal{H}_\infty$ condition. In the year 2013, the authors Phat and Trinh continue to study the $\mathcal{H}_\infty$ control problem for delayed neural networks

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-\tau_1(t))) + Bu(t) + C\omega(t),\\ z(t) &= Ex(t) + Mh(x(t-\tau_2(t))) + Nu(t),\end{aligned}$$

in two cases as follows: the delay functions $\tau_1(t)$, $\tau_2(t)$ are differentiable and their derivatives are bounded by a positive real number less than 1 or the delay functions are bounded but not necessarily differentiable. And then, the authors obtained the exponentially stabilizability and $\mathcal{H}_\infty$ condition for each case.

To sum up, the above-mentioned results for stability and $\mathcal{H}_\infty$ control are mainly restricted by the assumption that the delays are differentiable functions and that their derivatives have upper limit or simply bounded functions. Besides, studying the $\mathcal{H}_\infty$ control problem for the system (1) with $h(t)$ is a continuous function, no need for differentiability and taking value in a interval has not received adequate attention from researchers. In that context, we propose the $\mathcal{H}_\infty$ control problem for the system (1).

The second problem we are interested in this thesis is the finite-time $\mathcal{H}_\infty$ control problem for the linear discrete-time systems with interval time-varying delay:

$$\begin{aligned}x(k+1) &= Ax(k) + A_d x(k-d(k)) + Bu(k) + G\omega(k),\\ z(k) &= Cx(k) + C_d x(k-d(k)), \quad k \in \mathbb{Z}_+,\\ x(k) &= \varphi(k), \quad k \in \{-d_2, -d_2+1, \ldots, 0\},\end{aligned} \tag{2}$$

where $d(k)$ is a delay function satisfying: $0 < d_1 \leqslant d(k) \leqslant d_2 \quad \forall k \in \mathbb{Z}_+$. In 2010, the finite-time $\mathcal{H}_\infty$ control problem for the linear discrete-time

systems without delay

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + G\omega(k), \\ z(k) &= Cx(k) + D_1u(k) + D_2\omega(k), \end{aligned}$$

was proposed by Wang et al. This same problem for switched nonlinear discrete-time systems without delay was investigated by Xiang and Xiao in 2011. By 2012, Song et al. has taken one step further in solving this problem for linear discrete-time systems with constant delay

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}x(k) + A_{d,\sigma(k)}x(k-d) + B_{\sigma(k)}u(k) + G_{\sigma(k)}\omega(k), \\ z(k) &= C_{\sigma(k)}x(k) + C_{d,\sigma(k)}x(k-d) + D_{\sigma(k)}u(k) + F_{\sigma(k)}\omega(k). \end{aligned}$$

Shortly thereafter, this result was extended to switched nonlinear discrete-time systems with constant delay by Zong et al. (2015). On the finite-time stability and stabilization for linear discrete-time systems with interval time-varying delay

$$x(k+1) = Ax(k) + A_dx(k-d(k)) + Bu(k),$$

there are two fairly typical papers that were announced by Zuo et al. and Zhang et al. respectively in 2013 and 2014.

It is clear that the above results for finite-time $\mathcal{H}_\infty$ control for both linear and nonlinear discrete-time systems are restricted by the assumption that there is no presence of delay or there is the presence of delay but simply a constant function. The current study of the finite-time $\mathcal{H}_\infty$ control problem for the system (2) with the delay function $d(k)$ satisfies the above-mentioned interval time-varying condition have not received the attention of researchers. In this context, we propose the finite-time $\mathcal{H}_\infty$ control problem for the system (2).

The third problem to be addressed in this thesis is the finite-time $\mathcal{H}_\infty$ control problem for the singular discrete-time neural networks with interval time-varying delay:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Wf(x(k)) + W_1g(x(k-d(k))) + Bu(k) + C\omega(k), \\ z(k) &= A_1x(k) + Dx(k-d(k)) + B_1u(k), \quad k \in \mathbb{Z}_+, \\ x(k) &= \varphi(k), \quad k \in \{-d_2, -d_2+1, \ldots, 0\}, \end{aligned} \tag{3}$$

where delay function $d(k)$ are assumed to be time-varying and belong to a interval as in the system (2). The study of the $\mathcal{H}_\infty$ control problem for

the discrete-time neural networks with interval time-varying delays occured early with two articles Lu et al. (2009) and Sakthivel et al. (2012). However, the finite-time stability for this class of systems has only been recently investigated by some researchers. Specifically, the finite-time boundedness for discrete-time neural networks with time-varying delay was surveyed by Zhang et al. in 2014, while the finite-time stability for discrete-time fuzzy neural networks without delay was obtained by Bai et al. in 2015.

At present, the study of stability and control of singular systems are developing strongly in both theoretical and applied directions. We would like to point out the research situation for this class of systems as follows. Stability and robust stabilisation for a class of non-linear uncertain discrete-time descriptor Markov jump systems without delay were investigated by Song et al. in 2012. Very soon after, this result was further developed for the system with time-varying delay in Wang and Ma (2013). For the finite-time $\mathcal{H}_\infty$ control problem, the article series Zhang et al. (2014), Ma et al. (2015) and Ma et al. (2016) in that order considered this problem for linear discrete-time singular systems with no delay, constant delay, and time-varying delay. A model for discrete-time singular neural networks can be found in Hahanov and Rutkas (2009) and the stochastic stability for discrete singular neural networks with Markovian jump and mixed time-delays was introduced by Ma and Zheng in 2016.

To our knowledge, until now, the study of the finite-time $\mathcal{H}_\infty$ control problem for the systems (3) with the interval-like time-varying delay function $d(k)$ has not received the attention of researchers. In that context, we propose the finite-time $\mathcal{H}_\infty$ control problem for the systems (3).

3. PURPOSE, OBJECT AND SCOPE OF THE THESIS

The thesis focuses on the study of the construction of new Lyapunov–Krasovskii functionals in order to obtain new significant criteria that solve the $\mathcal{H}_\infty$ control problem for some classes of functional differential/difference equations with the extended delay structure and some classes of functional differential/difference equations have more general structure. Namely as follows:

• **Content 1:** To study the $\mathcal{H}_\infty$ control problem for the neural networks with mixed time-varying delays.

• **Content 2:** To study the finite-time $\mathcal{H}_\infty$ control problem for the linear discrete-time systems with interval time-varying delay.

• **Content 3:** To study the finite-time $\mathcal{H}_\infty$ control problem for the singular discrete-time neural networks with interval time-varying delay.

4. RESEARCH METHODS OF THE THESIS

In order to achieve that goal, the thesis develops the Lyapunov–Krasovskii functionals technique, in combination with some of the existing tools in analysis, linear algebra, ordinary differential equations and singular differential equations.

5. RESULTS OF THE THESIS

The thesis has achieved the following main results:

• Designing a feedback controller solves the $\mathcal{H}_\infty$ control problem for neural networks with mixed time-varying delays.

• Proposing sufficient conditions to ensure the $\mathcal{H}_\infty$ finite-time boundedness for linear discrete-time systems with interval time-varying delay. Then designing a feedback controller that solves the finite-time $\mathcal{H}_\infty$ control problem for this class of systems.

• The corresponding results for singular discrete-time neural networks with interval time-varying delay are also set. In addition, with this class of systems, we have simultaneously demonstrated regularity, causality, and unique existence of solution of the system in a neighborhood of the origin.

6. ARRANGEMENT OF THE THESIS

The thesis is arranged as follows. Apart from Introduction, Conclusion, List of Published Works and References, the thesis consists of four chapters: Chapter 1 systematically summarizes the preparation knowledge. Chapter 2 presents a finding of the $\mathcal{H}_\infty$ control problem for the neural networks with mixed time-varying delays. Chapter 3 presents findings of $\mathcal{H}_\infty$ finite-time boundedness and finite-time $\mathcal{H}_\infty$ control for the linear discrete-time systems with interval time-varying delay. Chapter 4 presents the solution of the finite-time $\mathcal{H}_\infty$ control problem for the singular discrete-time neural networks with interval time-varying delay together related results.

Chapter 1

PRELIMINARIES

This chapter presents briefly some of the classic results in theory of time-delay systems. The stability, stabilization and $\mathcal{H}_\infty$ control problem will in turn be presented along with some additional knowledge needed for the following chapters. The main contents of this chapter are extracted from Hien (2010), Thanh (2015), Gu et al. (2003), Hale et al. (1993), Kharitonov (2013), Kolmanovskii and Myshkis (1999), Wu et al. (2010), Zhang and Chen (1998) and Zhou et al. (1995).

1.1. Stability and stabilization problems for time-delay systems

1.1.1. Stability problem

In this section, we first present theorems of existence and uniqueness of local solutions and existence and uniqueness of global solutions of differential systems with delay; then we state concepts: stability, asymptotic stability, exponential stability, etc., along with the Lyapunov–Krasovskii criteria ensure the corresponding stability.

Next, we provide definitions of stability and asymptotic stability for difference systems with delay.

1.1.2. Stabilization problem

In this section, we first present definitions of stabilizability and $\alpha-$exponential stabilizability of control systems with delay.

The next is the presentation on stabilizability of discrete-time control systems with delay.

1.2. The $\mathcal{H}_\infty$ control problem

1.2.1. The $\mathcal{H}_\infty$ space

This section introduces definition of the $\mathcal{H}_\infty$ space and formula for the $\mathcal{H}_\infty$ norm of the transfer matrix function from ω to z.

1.2.2. The $\mathcal{H}_\infty$ control problem

This section is intended to discuss the optimal $\mathcal{H}_\infty$ control problem and the suboptimal $\mathcal{H}_\infty$ control problem.

1.3. Linear matrix inequality

Most of this section is dedicated to introducing the concept of linear matrix inequality (LMI) and the standard LMI problem. This section is closed with the famous Schur Complement Lemma, which is often used as an effective tool to transform nonlinear matrix inequalities into LMI.

Chapter 2

$\mathcal{H}_\infty$ CONTROL PROBLEM FOR NEURAL NETWORKS WITH MIXED TIME-VARYING DELAYS

This chapter aims to present the first result of the thesis. Specifically, we have proposed a sufficient condition for the $\mathcal{H}_\infty$ control problem for neural networks with mixed time-varying delays. The content of this chapter is written based on the paper [1] in the author's works related to the thesis that has been published.

2.1. STATEMENT OF THE PROBLEM

Consider a neural networks model is described by the differential system with mixed time-varying delays:

$$\begin{aligned}
\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2 \int_{t-k(t)}^{t} c(x(s))ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad + Bu(t) + C\omega(t) \\
z(t) &= Ex(t) + Mx(t-h(t)) + Nu(t), \quad t \geqslant 0, \\
x(t) &= \varphi(t), \quad t \in [-d, 0],
\end{aligned} \tag{2.1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\mathsf{T} \in \mathbb{R}^n$ is the state vector of the neural networks; $u(t) \in L^2([0,s], \mathbb{R}^m)$ $\forall s > 0$, is the control input; $\omega(t) \in L^2([0,\infty), \mathbb{R}^r)$ is the uncertain input; $z(t) \in \mathbb{R}^s$ is the observation output of the neural networks; the diagonal matrix $A = \mathrm{diag}\{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n\}, \overline{a}_i > 0$ $\forall i = \overline{1,n}$ represents the self-feedback term and the matrices $W_0, W_1, W_2 \in \mathbb{R}^{n\times n}$ denote, respectively, the connection weights, the delayed connection weights, the distributively delayed connection weights; $B \in \mathbb{R}^{n\times m}, N \in \mathbb{R}^{s\times m}$ denote the control input matrices; $C \in \mathbb{R}^{n\times r}$ denotes the perturbation/uncertain input matrix; $E, M \in \mathbb{R}^{s\times n}$ denotes the observation output matrix; the time-varying delay functions $h(t), k(t)$ satisfy the condition

$$0 \leqslant h_1 \leqslant h(t) \leqslant h_2, \quad 0 \leqslant k(t) \leqslant k,$$

where h_1, h_2, k are given constants. The initial function $\varphi(t) \in C^1([-d,0], \mathbb{R}^n)$ with $d = \max\{h_2, k\}$, and

$$\begin{aligned}
f(x(t)) &= [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^\mathsf{T}, \\
g(x(t-h(t))) &= [g_1(x_1(t-h(t))), g_2(x_2(t-h(t))), \ldots, g_n(x_n(t-h(t)))]^\mathsf{T},
\end{aligned}$$

$$c(x(t)) = [c_1(x_1(t)), c_2(x_2(t)), \ldots, c_n(x_n(t))]^{\mathsf{T}},$$

are various activation functions such that with every $i \in \{1, \ldots, n\}$, $f_i(\cdot)$, $g_i(\cdot)$ and $c_i(\cdot)$ are real one variable functions which are Lipschitz continuous with the corresponding Lipschitz constants a_i, b_i and c_i. Furthermore, assuming that $f_i(0) = g_i(0) = c_i(0) = 0 \ \ \forall i = \overline{1, n}$.

Remark 2.1. (i) The above assumptions imply directly for every $i \in \{1, \ldots, n\}$, the following growth conditions hold:

$$|f_i(\xi)| \leqslant a_i|\xi|, \quad |g_i(\xi)| \leqslant b_i|\xi|, \quad |c_i(\xi)| \leqslant c_i|\xi| \quad \forall \xi \in \mathbb{R}.$$

(ii) The neural networks model is described by the differential systems with mixed time-varying delays (2.1) with different activation functions $f(x(t))$, $g(x(t-h(t)))$, $c(x(t))$ satisfying the above Lipschitz condition will have a unique solution on $[0, +\infty)$ by Theorem 1.3, Chapter 1 of the thesis.

Definition 2.1. Given $\alpha > 0$. The zero solution of system (2.1), where $u \equiv 0$, $\omega \equiv 0$, is α-exponentially stable if there is a positive number $N > 0$ such that every solution of the system satisfies:

$$\|x(t, \varphi)\| \leqslant N\|\varphi\|_{C^1} e^{-\alpha t} \quad \forall t \geqslant 0.$$

Definition 2.2. Given $\alpha > 0$, $\gamma > 0$. The $\mathcal{H}_\infty$ control problem for system (2.1) in proportion to α, γ has a solution if there exists a memoryless state feedback controller $u(t) = Kx(t), K \in \mathbb{R}^{m \times n}$ satisfying the following two requirements:

(i) The zero solution of the closed-loop system:

$$\begin{aligned} \dot{x}(t) = -[A - BK]x(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) \\ + W_2 \int_{t-k(t)}^{t} c(x(s))ds \\ x(t) = \varphi(t), \ t \in [-d, 0], \end{aligned}$$

is α-exponentially stable.

(ii) There is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^\infty \|z(t)\|^2 dt}{c_0\|\varphi\|_{C^1}^2 + \int_0^\infty \|\omega(t)\|^2 dt} \leqslant \gamma, \tag{2.2}$$

where the supremum is taken over all $\varphi(t) \in C^1([-d,0],\mathbb{R}^n)$ and the non-zero uncertainty $\omega(t) \in L^2([0,\infty),\mathbb{R}^r)$, $\omega \not\equiv 0$.

In this case we say that the feedback $\mathcal{H}_\infty$ control $u(t) = Kx(t)$ exponentially stabilizes the system (2.1).

Remark 2.2. Recall that most often practical systems (including the neural control system) are subject to external disturbances and in some cases this can degrade performance if they are not taken into account during the design phase. Many approaches have been proposed to deal with this problem and one of them is the $\mathcal{H}_\infty$ control technique with the assumption that the disturbance belongs to $L^2[0,\infty)$. As discussed in Section 1.2, Chapter 1, the idea here is to design an suboptimal control to minimize the effects of the external disturbance on the output. In particular, design a controller that guarantees that $\mathcal{H}_\infty-$norm of the transfer function between the controlled output $z(t)$ and the external disturbance $\omega(t)$ will not exceed a given level $\gamma > 0$. From there, the relationship between the input and the output

$$\|z\|_2 \leqslant \gamma\|\omega\|_2 \quad \forall\omega \in L^2([0,\infty),\mathbb{R}^q)$$

is established at the end of Section 1.2.2, Chapter 1 in the context of no delay and initial condition $x(0) = 0$. Here, we proposed (2.2) as an extension of the above constraint that has the form

$$\|z\|_2^2 \leqslant \gamma(c_0\|\varphi\|_{C^1}^2 + \|\omega\|_2^2) \quad \forall\varphi(\cdot) \in C^1([-d,0],\mathbb{R}^n),\ \forall\omega(\cdot) \in L^2([0,\infty),\mathbb{R}^r),$$

for the purpose of evaluating the output error z depends on both the external disturbance ω and the initial condition φ of the state x.

2.2. THE MAIN RESULT

Before introducing main result, the notations of several matrix variables are defined for simplicity.

$$\begin{aligned}
&F = \mathrm{diag}\{a_1,\dots,a_n\},\ \ G = \mathrm{diag}\{b_1,\dots,b_n\},\ \ H = \mathrm{diag}\{c_1,\dots,c_n\},\\
&c^2 = \max\{c_1^2,\dots,c_n^2\},\\
&P_1 = P^{-1},\ \ Q_1 = P^{-1}QP^{-1},\ \ R_1 = P^{-1}RP^{-1},\ \ S_1 = P^{-1}SP^{-1},\\
&\alpha_1 = \lambda_{\min}(P_1),\\
&\alpha_2 = \lambda_{\max}(P_1) + h_1\lambda_{\max}(Q_1) + \frac{1}{2}h_2^3\lambda_{\max}(R_1)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(h_2-h_1)^2(h_2+h_1)\lambda_{\max}(S_1)+\frac{1}{2}c^2k^2\lambda_{\max}(D_2^{-1}),\\
\Omega_{11} &= -(AP+PA)+Q+2\alpha P+\frac{2}{\gamma}CC^{\mathsf{T}}+2ke^{2\alpha k}W_2D_2W_2^{\mathsf{T}}-\frac{3}{4}BB^{\mathsf{T}}\\
&\quad -e^{-2\alpha h_2}R+W_0D_0W_0^{\mathsf{T}}+W_1D_1W_1^{\mathsf{T}},\\
\Omega_{12} &= -PA-\frac{1}{2}BB^{\mathsf{T}},\\
\Omega_{22} &= -2P+h_2^2R+(h_2-h_1)^2S+2ke^{2\alpha k}W_2D_2W_2^{\mathsf{T}}+\frac{2}{\gamma}CC^{\mathsf{T}}\\
&\quad +W_0D_0W_0^{\mathsf{T}}+W_1D_1W_1^{\mathsf{T}},\\
\Omega_{33} &= -e^{-2\alpha h_1}Q-e^{-2\alpha h_2}S,\ \ \Omega_{44}=-e^{-2\alpha h_2}R-e^{-2\alpha h_2}S.
\end{aligned}
$$

For simplicity of expression as in Petersen et al. (2000), we assume that matrices E, M, N of the system (2.1) satisfy

$$N^{\mathsf{T}}[E\ \ M]=0,\quad N^{\mathsf{T}}N=I.$$

Theorem 2.1. *Given $\alpha>0$, $\gamma>0$. Suppose the matrix coefficients of the system (2.1) satisfy: there exist symmetric positive definite matrices P, Q, R, S and three diagonal positive definite matrices D_0, D_1, D_2 such that the following LMI holds:*

$$
\Omega=\begin{bmatrix}
\Omega_{11} & \Omega_{12} & 0 & e^{-2\alpha h_2}R & PE^{\mathsf{T}} & PF & PH & 0 & 0\\
* & \Omega_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & \Omega_{33} & e^{-2\alpha h_2}S & 0 & 0 & 0 & 0 & 0\\
* & * & * & \Omega_{44} & 0 & 0 & 0 & PM^{\mathsf{T}} & PG\\
* & * & * & * & -\frac{1}{2}I & 0 & 0 & 0 & 0\\
* & * & * & * & * & -\frac{1}{2}D_0 & 0 & 0 & 0\\
* & * & * & * & * & * & -\frac{1}{k}D_2 & 0 & 0\\
* & * & * & * & * & * & * & -\frac{1}{2}I & 0\\
* & * & * & * & * & * & * & * & -\frac{1}{2}D_1
\end{bmatrix}<0. \tag{2.3}
$$

Then, the $\mathcal{H}_\infty$ control problem of system (2.1) in proportion to α,γ has a solution. Moreover, stabilizing feedback controller is

$$u(t)=-\frac{1}{2}B^{\mathsf{T}}P^{-1}x(t),\quad t\geqslant 0,$$

and the solution of the system, when $\omega\equiv 0$, satisfies

$$\|x(t,\varphi)\|\leqslant\sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{C^1}e^{-\alpha t}\quad \forall t\geqslant 0.$$

Remark 2.3. In the papers He et al. (2007), Kwon and Park (2009), Sakthivel et al. (2012), additional unknowns and free-weighting matrices were introduced to provide flexibility in solving the obtained LMIs. However, too many unknowns and free-weighting matrices employed in the existing methods complicate the system analysis and significantly increase the computational demand. In order to avoid that disadvantage, Theorem 2.1 does not involve such free-weighting matrices.

Remark 2.4. Our proposed results have also overcome the limitations of existing results (He et al. (2007), Phat and Trinh (2010), Phat and Trinh (2013), Sakthivel et al. (2012).) of differentiability of delays; moreover, the discrete delay $h(t)$ has also been extended successfully to the case of taking value in a interval, i.e., the lower bound of $h(t)$ may be a positive real number. In addition, the stabilization controller are designed based on the solution finding of a LMI. For that reason, our criterion is a significant expansion of the criteria recommended in Phat and Trinh (2013), Sakthivel et al. (2012).

Remark 2.5. It is clear that the terms of the Ω block matrix depend monotonically on delays so the feasibility of LMI (2.3) will increase as the quantities h_1, h_2, k become smaller. In particular, if (2.3) has a solution with some positive h_1, h_2, k delays, then it will also feasible for all $\bar{h}_1, \bar{h}_2, \bar{k}$ which are less than h_1, h_2, k in that order.

2.3. ILLUSTRATION

In this section, we provide a numerical example to illustrate the effectiveness of the obtained conditions in Theorem 2.1.

Chapter 3

FINITE-TIME $\mathcal{H}_\infty$ CONTROL PROBLEM FOR LINEAR DISCRETE-TIME SYSTEMS WITH INTERVAL TIME-VARYING DELAY

This chapter aims to present the sufficient conditions for solving the finite-time $\mathcal{H}_\infty$ control problem for linear discrete-time systems with interval time-varying delay, which is also the second result we got in the process of doing the topic. The content of this chapter is written based on the paper [2] in the author's works related to the thesis that has been published.

3.1. CONCEPT OF FINITE-TIME STABILITY

In this section, we present the concept of finite-time stability and assert its independence with the concept of Lyapunov stability.

3.2. STATEMENT OF THE PROBLEM

Consider the following linear discrete-time systems with interval time-varying delay:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-h(k)) + Bu(k) + G\omega(k), \\ z(k) &= Cx(k) + C_d x(k-h(k)), \quad k \in \mathbb{Z}_+, \\ x(k) &= \varphi(k), \quad k \in \{-h_2, -h_2+1, \ldots, 0\}, \end{aligned} \tag{3.1}$$

where $x(k) \in \mathbb{R}^n$ is the state; $u(k) \in \mathbb{R}^m$ is the control input; $z(k) \in \mathbb{R}^p$ is the observation output; $A, A_d \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, G \in \mathbb{R}^{n\times q}, C, C_d \in \mathbb{R}^{p\times n}$ are given real constant matrices; $h(k)$ is a delay function satisfying

$$0 < h_1 \leqslant h(k) \leqslant h_2 \quad \forall k \in \mathbb{Z}_+,$$

where h_1, h_2 are known positive integers; $\varphi(k)$ is the initial function; $\omega(k) \in \mathbb{R}^q$ satisfying the condition

$$\sum_{k=0}^{N} \omega^{\mathsf{T}}(k)\omega(k) < d, \tag{3.2}$$

with d is a given positive real number.

Definition 3.1. Given positive numbers $N, c_1, c_2, c_1 < c_2$ and a symmetric positive definite matrix R, discrete-time delay system (3.1) with $u(k) = 0$ is

said to be finite-time bounded w.r.t. (c_1, c_2, R, N) if

$$\max_{k\in\{-h_2,-h_2+1,\ldots,0\}} \varphi^{\mathsf{T}}(k)R\varphi(k) \leqslant c_1 \implies x^{\mathsf{T}}(k)Rx(k) < c_2 \ \ \forall k = \overline{1, N},$$

for all disturbances $\omega(k)$ satisfying (3.2).

Definition 3.2. Given positive numbers $\gamma, N, c_1, c_2, c_1 < c_2$ and a symmetric positive definite matrix R, system (3.1) with $u(k) = 0$ is said to be $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N) if the following two conditions hold:

(i) System (3.1) with $u(k) = 0$ is finite-time bounded w.r.t. (c_1, c_2, R, N).

(ii) Under the zero initial condition (i.e. $\varphi(k) = 0 \ \forall k \in \{-h_2, -h_2 + 1, \ldots, 0\}$), the output $z(k)$ satisfies

$$\sum_{k=0}^{N} z^{\mathsf{T}}(k)z(k) \leqslant \gamma \sum_{k=0}^{N} \omega^{\mathsf{T}}(k)\omega(k), \tag{3.3}$$

for all disturbances $\omega(k)$ satisfying (3.2).

Definition 3.3. Given positive numbers $\gamma, N, c_1, c_2, c_1 < c_2$, and a symmetric positive definite matrix R, the finite-time $\mathcal{H}_\infty$ control problem for system (3.1) has a solution if there exists a state feedback controller $u(k) = Kx(k)$ such that the resulting closed-loop system is $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N).

3.3. THE MAIN RESULTS

Theorem 3.1. *Given positive constants γ, N, c_1, c_2 with $c_1 < c_2$, a symmetric positive definite matrix R. Suppose the matrix coefficients of the system (3.1) satisfy: there exist symmetric positive-definite matrices P, Q, positive scalars $\lambda_1, \lambda_2, \lambda_3$ and a scalar $\delta \geqslant 1$ such that the following matrix inequalities hold:*

$$\lambda_1 R < P < \lambda_2 R, \quad Q < \lambda_3 R, \tag{3.4}$$

$$\begin{bmatrix} -\delta P + (h_2 - h_1 + 1)Q & 0 & 0 & A^{\mathsf{T}}P & C^{\mathsf{T}} \\ * & -\delta^{h_1}Q & 0 & A_d^{\mathsf{T}}P & C_d^{\mathsf{T}} \\ * & * & -\frac{\gamma}{\delta^N}I & G^{\mathsf{T}}P & 0 \\ * & * & * & -P & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{3.5}$$

$$\begin{bmatrix} \gamma d - c_2\delta\lambda_1 & c_1\delta^{N+1}\lambda_2 & \rho\lambda_3 \\ * & -c_1\delta^{N+1}\lambda_2 & 0 \\ * & * & -\rho\lambda_3 \end{bmatrix} < 0. \tag{3.6}$$

Then system (3.1) with $u(k) = 0$ is $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N), where

$$\rho := c_1\delta^{N+h_2-1}\Big[h_2\delta + \frac{h_2(h_2-1) - h_1(h_1-1)}{2}\Big].$$

Theorem 3.2. *Given positive constants γ, N, c_1, c_2 with $c_1 < c_2$, a symmetric positive-definite matrix R. Suppose the matrix coefficients of the system (3.1) satisfy: there exist symmetric positive-definite matrices U, V, W_1, W_2, W_3, a matrix Y and a scalar $\delta \geqslant 1$ such that the following matrix inequalities hold:*

$$U < W_2, \quad V < W_3, \tag{3.7}$$

$$\begin{bmatrix} -\delta U + (h_2 - h_1 + 1)V & 0 & 0 & UA^\mathsf{T} + Y^\mathsf{T}B^\mathsf{T} & UC^\mathsf{T} \\ * & -\delta^{h_1}V & 0 & UA_d^\mathsf{T} & UC_d^\mathsf{T} \\ * & * & -\frac{\gamma}{\delta^N}I & G^\mathsf{T} & 0 \\ * & * & * & -U & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{3.8}$$

$$\begin{bmatrix} -W_1 & c_1\delta^{N+1}W_2 & \rho W_3 \\ * & -c_1\delta^{N+1}W_2 & 0 \\ * & * & -\rho W_3 \end{bmatrix} < 0, \tag{3.9}$$

$$\begin{bmatrix} W_1 - c_2\delta U & \gamma dUR \\ * & -\gamma dR \end{bmatrix} < 0. \tag{3.10}$$

Then the finite-time $\mathcal{H}_\infty$ control of system (3.1) has a solution. Moreover, the state feedback controller is given by

$$u(k) = YU^{-1}x(k), \quad k \in \mathbb{Z}_+.$$

Remark 3.1. As in the papers Zong et al. (2015) and Zuo et al. (2013), in order to prove Theorem 3.1 (and after that Theorem 3.2), we sought to construct a new set of Lyapunov–Krasovskii-type functionals which involved the coefficients δ^{k-1-s} and δ^{k-1-t}. By that way, we have avoided the need to transform the original system into two subsystems, as the authors have done in Zhang et al. (2014) and the obtained conditions (3.4)-(3.6) of Theorem 3.1 and (3.7)-(3.10) of Theorem 3.2 are also in terms of LMIs as in Zhang et

al. (2014). Here, the δ parameter plays the role as a adjustable parameter and (3.5)-(3.6), (3.8)-(3.10) will become LMIs when we fix this δ parameter, so they can be programmed and calculated easily by using the LMI toolbox in MATLAB. This is also a remarkable advantage of these two theorems of us in comparison with: conditions (29), (39) in Song et al. (2012), conditions (45), (56) in Zong et al. (2015) and condition (5) in Zuo et al. (2013).

Remark 3.2. In the articles: He et al. (2008), Liu et al. (2011), Song et al. (2012), Xiang and Xiao (2011), additional unknowns and free-weighting matrices are introduced to provide flexibility in solving the obtained LMI. However, too many unknowns and free-weighting matrices employed in the existing methods complicate the system analysis and significantly increase the computational demand. Compared with the free matrix method was used by that authors, our method uses fewer variables, for example, the LMI (3.5) has no free-weighting matrices, LMI (3.8) has only one free-weighting matrix. Therefore, the conditions we propose are less conservative than those mentioned above.

3.4. ILLUSTRATION

In this section, we provide two numerical examples to illustrate effectiveness of the achieved conditions in the Theorem 3.1 and Theorem 3.2, respectively.

Chapter 4

FINITE-TIME $\mathcal{H}_\infty$ CONTROL PROBLEM FOR SINGULAR DISCRETE-TIME NEURAL NETWORKS WITH INTERVAL TIME-VARYING DELAY

The third result of the thesis will be presented in this chapter. Specifically, we will mention the conditions that solve the finite-time $\mathcal{H}_\infty$ control problem for the singular discrete-time neural networks with interval time-varying delay. The content of the chapter is extracted from the article [3] in the list of published scientific works of the author related to the thesis.

4.1. A SKETCH OF LINEAR SINGULAR DISCRETE-TIME SYSTEMS

In this section, we present a sketch of the regularity and causality of the linear singular discrete-time systems with constant delay:

$$\begin{aligned} Ex(k+1) &= A_0x(k) + A_1x(k-\tau) + Bu(k), \quad k \in \mathbb{Z}_+, \\ x(k) &= \varphi(k), \quad k \in \{-\tau, -\tau+1, \dots, 0\}. \end{aligned} \tag{4.1}$$

One remarkable result is that with any compatible initial function $\varphi(k)$, from regularity and causality of the linear system (4.1), we can assert that the system has unique solution.

4.2. STATEMENT OF THE PROBLEM

Consider the following discrete-time singular neural networks with time-varying delay:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Wf(x(k)) + W_1g(x(k-h(k))) + Bu(k) + C\omega(k), \\ z(k) &= A_1x(k) + Dx(k-h(k)) + B_1u(k), \quad k \in \mathbb{Z}_+, \\ x(k) &= \varphi(k), \quad k \in \{-h_2, -h_2+1, \dots, 0\}, \end{aligned} \tag{4.2}$$

where $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^\mathsf{T} \in \mathbb{R}^n$ is the neuron state vector; n is the number of neurals; $u(k) \in \mathbb{R}^m$ is the control input; $z(k) \in \mathbb{R}^p$ is the observation output of the neural networks;

$$\begin{aligned} f(x(k)) &= [f_1(x_1(k)), f_2(x_2(k)), \dots, f_n(x_n(k))]^\mathsf{T}, \\ g(x(k-h(k))) &= [g_1(x_1(k-h(k))), g_2(x_2(k-h(k))), \dots, g_n(x_n(k-h(k)))]^\mathsf{T} \end{aligned}$$

are various neural activation functions, where f_i, g_i, $i = \overline{1,n}$, are continuously differentiable in a neighbourhood of the origin and satisfy the following

growth conditions: for every $i \in \{1, \ldots, n\}$, exist a_i, b_i such that:

$$|f_i(\xi)| \leqslant a_i|\xi|, \quad |g_i(\xi)| \leqslant b_i|\xi| \quad \forall \xi \in \mathbb{R}.$$

$E \in \mathbb{R}^{n\times n}$ be singular matrix and $\operatorname{rank}(E) = r \leqslant n$. The diagonal matrix $A = \operatorname{diag}\{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n\}$, $|\overline{a}_i| < 1$ $\forall i = \overline{1, n}$ represents the self-feedback term; the matrices W, $W_1 \in \mathbb{R}^{n\times n}$ denote the connection weight matrix and the delayed connection weight matrix, respectively; $B \in \mathbb{R}^{n\times m}$, $B_1 \in \mathbb{R}^{p\times m}$ are the control input matrices; $C \in \mathbb{R}^{n\times q}$ is the perturbation/uncertain input matrix; $A_1, D \in \mathbb{R}^{p\times n}$ denote the observation output matrix; the time-varying delay functions $h(k)$ satisfies the condition

$$0 < h_1 \leqslant h(k) \leqslant h_2 \quad \forall k \in \mathbb{Z}_+,$$

where h_1, h_2 are given positive integers; $\varphi(k)$ is the initial function; the external disturbance $\omega(k) \in \mathbb{R}^q$ satisfying the condition

$$\sum_{k=0}^{N} \omega^{\mathsf{T}}(k)\omega(k) < d,$$

where $d > 0$ is a given scalar.

Definition 4.1. The pair (E, A) is said to be regular if characteristic polynomial $\det(sE - A)$, where $s \in \mathbb{C}$, is not identical zero. The pair (E, A) is said to be causal if $\deg(\det(sE - A)) = \operatorname{rank}(E)$. System (4.2) with $u(k) = 0$ is said to be regular and causal if the pair (E, A) is regular and causal.

Remark 4.1. If the pair (E, A) is regular and causal, a singular system can be partitioned into two parts, namely a dynamic subsystem and an algebraic constraint (see Dai (1989), Sau (2018)). If an initial condition satisfies an algebraic constraint, then the initial condition is called a compatible initial condition. In contrast to the results given in the previous section for linear singular system, Example 1 in Lu et al. (2011) shows that even if the matrix pair (E, A) is regular and causal, solution of a nonlinear singular system may not exist with any compatible initial condition $x(0)$. In conclusion, the existence of solution is a fundamental problem for the nonlinear singular systems in general and singular neural networks in particular, and is completely independent of regularity and causality. For this reason, whenever we study class of this systems, existence and uniqueness of the solution, regularity and causality should be taken into account simultaneously.

Remark 4.2. The concepts of *finite-time boundedness* and $\mathcal{H}_\infty$ *finite-time boundedness w.r.t.* (c_1, c_2, R, N) *of the system (4.2) with* $u(k) = 0$ are completely defined in the same way as those given to the system (3.1) with $u(k) = 0$ (see the Definition 3.1 and Definition 3.2). The *finite time* $\mathcal{H}_\infty$ *control problem for the system (4.2)* is also defined quite similarly to the system (3.1) (see the Definition 3.3).

4.3. THE MAIN RESULTS

Consider the discrete-time singular neural networks (4.2) with $u(k) = 0$, due to $\operatorname{rank}(E) = r \leqslant n$ there are two nonsingular matrices M, $G \in \mathbb{R}^{n\times n}$ such that $MEG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Let

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} M, \quad MAG = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$M^{-\mathsf{T}} P M^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad F = \operatorname{diag}\{a_1, \ldots, a_n\}, \quad H = \operatorname{diag}\{b_1, \ldots, b_n\},$$

$$\Phi_{11} = -\delta E^{\mathsf{T}} P E + (h_2 - h_1 + 1)Q + S_1 + A_1^{\mathsf{T}} A_1 + F^2 - P\bar{M}A - A\bar{M}^{\mathsf{T}} P,$$

$$\Phi_{22} = \delta^{h_1}(-S_1 + S_2), \quad \Phi_{44} = -\delta^{h_1} Q + D^{\mathsf{T}} D + H^2.$$

From there, regularity, causality, uniqueness and existence of the solution of system (4.2) are guaranteed by the following theorem.

Theorem 4.1. *Given positive constants* γ, N. *Suppose the matrix coefficients of the system (4.2) satisfy: there exist symmetric positive-definite matrices* P, Q, S_1, S_2 *and a scalar* $\delta \geqslant 1$ *such that the following matrix inequality holds:*

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & 0 & A_1^{\mathsf{T}} D & -P\bar{M}W & -P\bar{M}W_1 & -P\bar{M}C & AP \\ * & \Phi_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\delta^{h_2} S_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & W^{\mathsf{T}} P \\ * & * & * & * & * & -I & 0 & W_1^{\mathsf{T}} P \\ * & * & * & * & * & * & -\frac{\gamma}{\delta^N} I & C^{\mathsf{T}} P \\ * & * & * & * & * & * & * & -P \end{bmatrix} < 0. \tag{4.3}$$

Then system (4.2) with $u(k) = 0$ *is regular, causal and has unique solution in a neighborhood of the origin.*

Remark 4.3. In Lu et al. (2011) (Song et al. (2012), respectively), the authors propose a sufficient condition for the existence and uniqueness of the solution of nonlinear singular discrete-time systems using fixed point principle (the implicit function theorem, respectively). In Theorem 4.1, by applying the implicit function theorem as in Song et al. (2012), we obtained a sufficient condition for not only the existence and uniqueness of the solution in a neighborhood of the origin, but also the regularity and causality of the system (4.2). Because the obtained condition is in terms of a matrix inequality, it can be effectively solved using the LMI toolbox in Matlab (see Gahinet et al. (1995)).

Next we state a sufficient condition to ensure that the system (4.2) with $u(k) = 0$ is $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N).

Theorem 4.2. *Given positive constants γ, N, c_1, c_2 with $c_1 < c_2$ and a symmetric positive-definite matrix R. Suppose the matrix coefficients of the system (4.2) satisfy: there exist symmetric positive-definite matrices P, Q, S_1, S_2, positive scalars λ_i, $i = \overline{1,5}$ and a scalar $\delta \geqslant 1$ such that the following matrix inequalities hold:*

$$\Psi = \left[\Psi_{ij}\right]_{11\times 11} < 0, \tag{4.4}$$

$$E^\mathsf{T} P E < \lambda_1 R, \quad Q < \lambda_2 R, \quad \lambda_3 R < S_1 < \lambda_4 R, \quad S_2 < \lambda_5 R, \tag{4.5}$$

$$\begin{bmatrix} \gamma d - c_2\lambda_3 & c_1\delta^{N+1}\lambda_1 & \rho\lambda_2 & c_1\delta^{N+h_1}h_1\lambda_4 & c_1\delta^{N+h_2}(h_2-h_1)\lambda_5 \\ * & -c_1\delta^{N+1}\lambda_1 & 0 & 0 & 0 \\ * & * & -\rho\lambda_2 & 0 & 0 \\ * & * & * & -c_1\delta^{N+h_1}h_1\lambda_4 & 0 \\ * & * & * & * & -c_1\delta^{N+h_2}(h_2-h_1)\lambda_5 \end{bmatrix} < 0. \tag{4.6}$$

Then system (4.2) with $u(k) = 0$ is $\mathcal{H}_\infty$ finite-time bounded w.r.t. (c_1, c_2, R, N), where

$$\begin{aligned}
&\Psi_{11} = -\delta E^\mathsf{T} P E + (h_2 - h_1 + 1)Q + S_1 - P\bar{M}A - A\bar{M}^\mathsf{T} P,\\
&\Psi_{15} = -P\bar{M}W, \;\; \Psi_{16} = -P\bar{M}W_1, \;\; \Psi_{17} = -P\bar{M}C, \;\; \Psi_{18} = AP,\\
&\Psi_{19} = A_1^\mathsf{T}, \;\; \Psi_{1,10} = F, \;\; \Psi_{22} = \Phi_{22}, \;\; \Psi_{33} = -\delta^{h_2}S_2, \;\; \Psi_{44} = -\delta^{h_1}Q,\\
&\Psi_{49} = D^\mathsf{T}, \;\; \Psi_{4,11} = H, \;\; \Psi_{55} = \Psi_{66} = \Psi_{99} = \Psi_{10,10} = \Psi_{11,11} = -I,\\
&\Psi_{58} = W^\mathsf{T}P, \;\; \Psi_{68} = W_1^\mathsf{T}P, \;\; \Psi_{77} = -\frac{\gamma}{\delta^N}I, \;\; \Psi_{78} = C^\mathsf{T}P, \;\; \Psi_{88} = -P,\\
&\Psi_{ij} = 0 \;\; \textit{forall other } i,\, j\!: j > i, \;\; \Psi_{ij} = \Psi_{ji}^\mathsf{T} \;\; \forall i, j : \; i > j,
\end{aligned}$$

$$\rho = c_1 \tfrac{h_2(h_2+1)-h_1(h_1-1)}{2} \delta^{N+h_2}.$$

Theorem 4.3. *Given positive constants γ, N, c_1, c_2 with $c_1 < c_2$ and a symmetric positive-definite matrix R. Suppose the matrix coefficients of the system (4.2) satisfy: there exist symmetric positive-definite matrices U_i, V_j with $i = \overline{1,4}$, $j = \overline{1,5}$, a matrix Y and a scalar $\delta \geqslant 1$ such that the following matrix inequalities hold:*

$$\Omega = \Big[\Omega_{ij}\Big]_{11\times 11} < 0, \tag{4.7}$$

$$\begin{bmatrix} -V_1 & U_1E^{\mathsf{T}} \\ * & -U_1 \end{bmatrix} < 0, \tag{4.8}$$

$$U_2 < V_2, \quad U_3 < V_4, \quad U_4 < V_5, \tag{4.9}$$

$$\begin{bmatrix} -V_3 & c_1\delta^{N+1}V_1 & \rho V_2 & c_1\delta^{N+h_1}h_1V_4 & c_1\delta^{N+h_2}(h_2-h_1)V_5 \\ * & -c_1\delta^{N+1}V_1 & 0 & 0 & 0 \\ * & * & -\rho V_2 & 0 & 0 \\ * & * & * & -c_1\delta^{N+h_1}h_1V_4 & 0 \\ * & * & * & * & -c_1\delta^{N+h_2}(h_2-h_1)V_5 \end{bmatrix} < 0, \tag{4.10}$$

$$\begin{bmatrix} V_3 - c_2U_3 & \gamma d U_1 R \\ * & -\gamma d R \end{bmatrix} < 0. \tag{4.11}$$

Then the finite-time $\mathcal{H}_\infty$ control of system (4.2) has a solution. Moreover, the state feedback controller is given by

$$u(k) = YU_1^{-1}x(k), \quad k \in \mathbb{Z}_+,$$

where

$$\begin{aligned}
\Omega_{11} &= \delta U_1 + (h_2 - h_1 + 1)U_2 + U_3 + \delta(U_1E^{\mathsf{T}} + EU_1) - \bar{M}(AU_1 + BY) \\
&\quad - (U_1A + Y^{\mathsf{T}}B^{\mathsf{T}})\bar{M}^{\mathsf{T}}, \\
\Omega_{15} &= -\bar{M}W, \;\; \Omega_{16} = -\bar{M}W_1, \;\; \Omega_{17} = -\bar{M}C, \;\; \Omega_{18} = U_1A + Y^{\mathsf{T}}B^{\mathsf{T}}, \\
\Omega_{19} &= U_1A_1^{\mathsf{T}} + Y^{\mathsf{T}}B_1^{\mathsf{T}}, \;\; \Omega_{1,10} = U_1F, \;\; \Omega_{22} = \delta^{h_1}(-U_3 + U_4), \\
\Omega_{33} &= -\delta^{h_2}U_4, \;\; \Omega_{44} = -\delta^{h_1}U_2, \;\; \Omega_{49} = U_1D^{\mathsf{T}}, \;\; \Omega_{4,11} = U_1H, \\
\Omega_{55} &= \Omega_{66} = \Omega_{99} = \Omega_{10,10} = \Omega_{11,11} = -I, \;\; \Omega_{58} = W^{\mathsf{T}}, \;\; \Omega_{68} = W_1^{\mathsf{T}}, \\
\Omega_{77} &= -\frac{\gamma}{\delta^N}I, \;\; \Omega_{78} = C^{\mathsf{T}}, \;\; \Omega_{88} = -U_1, \;\; \Omega_{ij} = 0 \;\; \textit{forall other } i, j\textit{: } j > i, \\
\Omega_{ij} &= \Omega_{ji}^{\mathsf{T}} \;\; \forall i, j: \; i > j, \;\; \rho = c_1 \tfrac{h_2(h_2+1)-h_1(h_1-1)}{2} \delta^{N+h_2}.
\end{aligned}$$

Remark 4.4. The results we got in the Theorem 4.2 and Theorem 4.3 can be considered as extensions of the results in Lu et al. (2009) and Ma and Zheng (2018) to the case $\mathcal{H}_\infty$ control for the singular discrete-time neural networks (4.2). To our knowledge, this is the first time that the finite-time $\mathcal{H}_\infty$ control problem for singular discrete-time neural networks with time-varying delay is considered. However, note that unlike most other papers in type of "the first time", where criteria are usually given in terms of delay-independent, all our criteria are delay-dependent, namely depend on the upper and lower bounds of the delay.

Remark 4.5. In the Theorems 4.1 - 4.3, we sought to construct a new set of Lyapunov–Krasovskii-type functionals which involved the coefficients δ^{k-1-s} and δ^{k-1-t}. Here, the δ parameter plays the role as a adjustable parameter and (4.3), (4.4), (4.6), (4.7) & (4.10) will become LMIs when we fix this δ parameter; therefore, they can be easily handled using the LMI toolbox in MATLAB. This is also a remarkable advantage of our theorems in comparison with: condition (31) in Ma et al. (2015), conditions (31), (40) & (49) in Ma et al. (2016) and condition (22b) in Zhang et al. (2014).

4.4. ILLUSTRATION

In this section, we provide two numerical examples to illustrate effectiveness of the obtained conditions in the Theorem 4.2 and Theorem 4.3, respectively.

CONCLUSION AND RECOMMENDATION

1. Results of the thesis

In this thesis, we have studied the $\mathcal{H}_\infty$ control problem for some classes of differential/difference systems with time-varying delay and get values in an interval. The obtained main results are

• Designing a feedback controller solves the $\mathcal{H}_\infty$ control problem for neural networks with mixed time-varying delays.

• Proposing sufficient conditions to ensure the $\mathcal{H}_\infty$ finite-time boundedness for linear discrete-time systems with interval time-varying delay. Then designing a feedback controller that solves the finite-time $\mathcal{H}_\infty$ control problem for this class of systems.

• The corresponding results for singular discrete-time neural networks with interval time-varying delay are also set. In addition, with this class of systems, we have simultaneously demonstrated regularity, causality, and unique existence of solution of the system in a neighborhood of the origin.

2. Recommendation of some further research issues

Besides the results achieved in the thesis, some open issues can be further studied such as:

• Study of stability and the (finite-time) $\mathcal{H}_\infty$ control problem for other classes of differential/difference systems and control systems such as nonlinear systems, switched systems, systems with Markovian jumps, etc. with interval time-varying delays.

• Study of (finite-time) stability and the design of other controller types, such as output feedback controllers, for classes of differential/difference systems and control systems with interval time-varying delays.

AUTHOR'S WORKS RELATED TO THE THESIS THAT HAVE BEEN PUBLISHED

[1] Le A. Tuan, Phan T. Nam and Vu N. Phat (2013), New $\mathcal{H}_\infty$ controller design for neural networks with interval time-varying delays in state and observation, *Neural Processing Letters*, Volume 37, Issue 3, 235-249. (SCIE)

[2] Le A. Tuan and Vu N. Phat (2016), Finite-time stability and $\mathcal{H}_\infty$ control of linear discrete-time delay systems with norm-bounded disturbances, *Acta Mathematica Vietnamica*, Volume 41, Number 3, 481-493. (SCOPUS)

[3] Le A. Tuan and Vu N. Phat (2018), Existence of solutions and finite-time stability for nonlinear singular discrete-time neural networks, *Bulletin of the Malaysian Mathematical Sciences Society*, Published Online: 13 February 2018, DOI: https://doi.org/10.1007/s40840-018-0608-y. (SCIE)

Results of the thesis have been reported at

- 8th National Mathematics Congress, Nha Trang, 08/2013.
- 13th Workshop on Optimization and Scientific Computing, Ba Vi, 23-25/04/2015.
- Seminar of Department of Analysis, Faculty of Mathematics - Informatics, Hanoi National University of Eduacation.
- Seminar of Department of Optimization and Control Theory, Institute of Mathematics, VAST.